A SEMI DISCRETE DYNAMICAL SYSTEM FOR A 2D DISSIPATIVE QUASI GEOSTROPHIC EQUATION

M. MOALLA-TRABELSI and E. ZAHROUNI
Unité de recherche : Multi-Fractals et Ondelettes
Faculté des Sciences de Monastir
Av. de l'environnemnt, Monastir
Tunisia

June 28, 2011

Abstract

A semi-discretization in time, according to a full implicit Euler scheme, for a 2D dissipative quasi geostrophic equation, is studied. We prove existence, uniqueness and regularity results of the solution to the predicted discretization, in the subcritical case for any initial data in $\dot{L}^2$. Hence, we define an infinite semi-discrete dynamical system, then we prove the existence and the regularity of the corresponding global attractor, for a source term f in $\dot{L}^{p_\alpha}$, for a fixed $p_\alpha = \frac{2}{1-\alpha}$.

1 Introduction

In this paper, we focus on a two dimensional dissipative quasi-geostrophic equation (QG),

$$\partial_t \theta + \nu(-\Delta)^\alpha \theta + u.\nabla\theta = f. \tag{1}$$

The solution θ of (1) is a real valued function defined on $\mathbb{R}_+ \times \Omega$, where Ω is either $\mathbb{R}^2$ or $\mathbb{T}^2 =]0, 2\pi[^2$. We assume that θ satisfies the following initial condition:

$$\theta(0, x) = \theta_0(x). \tag{2}$$

The solution θ represents the temperature of the fluid and $u = (u_1, u_2)$ is the divergence free velocity field which is related to θ by the mean of Riesz transforms according to:

$$u \ = \ \mathcal{R}^\perp(\theta) \ = \ (-\Lambda^{-1}\mathcal{R}_2\theta, \Lambda^{-1}\mathcal{R}_1\theta) = (-\partial_{x_2}(-\Delta)^{-\frac{1}{2}}\theta, \partial_{x_1}(-\Delta)^{-\frac{1}{2}}\theta). \tag{3}$$

The source term f is at least square integrable and time independent. $\nu > 0$ is the viscosity coefficient, and $\alpha \in (0, 1)$ is a fixed parameter. In the case where $\Omega = \mathbb{T}^2$, we suppose that

$$\theta \quad \text{is } 2\pi \text{ periodic in each direction.}$$

We notice that the case $\alpha = \frac{1}{2}$ is the dimensionally correct analogue of the 3D Navier-Stokes equation, this case is therefore called the critical one. Then $\alpha > \frac{1}{2}$ is called the subcritical case and $0 < \alpha < \frac{1}{2}$ is the supercritical one. These models arise under the assumptions of fast rotation, uniform stratification and uniform potential vorticity. The reader is referred to Canstantin, Majda and Tabak [7], Held and *al* [10], Pedolsky [15] and the references therein for more details.

Nowadays, there are intensive investigation about existence, uniqueness and regularity of solutions of (1) in the continuous case, for different values of the diffusion parameter α. Indeed, since the pioneering work of S. Resnick in [16], where weak solutions have been constructed, we can cite the works of Constantin Córdoba and Wu [5] in the critical case, and the one of Chae and Lee [4], in both critical and supercritical cases.
Let us mention also the work of Kiselev, Nazarov, and Volberg [14], and the work of Cafarelli and Vasseur [3], in the same direction.
We focus here on the subcritical case. In this framework, we refer the reader essentially to the paper of Constantin and Wu [8], where the authors showed that any solution with smooth initial value is smooth for all time. On the other hand, long time behavior of solutions of (1), was studied by N. Ju in [11] and by Berselli in [2]. These results give the proof of the existence of a global attractor, for the semi-group generated by the solutions of the quasi-geostrophic equations. An interesting question, is whether or not this important dynamical behavior, can be captured by some of the classical numerical schemes for solving (QG).

In fact, from the numerical point of view, Constantin and *al* in [6], performed a careful numerical study of the long time behavior of solutions to (QG).

To the best of our Knowledge, numerical schemes for solving (1) are seldom studied or even non-existent. Then, we focus in this paper, on a semi-discretization in time of (1) according to a full implicit Euler scheme, keeping the space variable continuous. In particular, we opted for this commonly used Euler scheme, since it is known to be a first order convergence scheme, unconditionally stable. Then, this scheme seems to be a suitable and reliable one to give answer to our expectations.

Here we are concerned with the discrete dynamical system associated to (5). More precisely, we prove firstly existence uniqueness and regularity of the solution to (5), and secondly we prove the existence of the global attractor.

We notice that N. Ju considered in [12], a time discretization of the non-stationary viscous incompressible Navier-Stokes equations, according to a linear backward Euler scheme. He treated either the no-slip boundary condition or the periodic boundary condition, in a 2D bounded domain, with a non-zero external force. As one of the main results obtained in [12], the global attractor for the approximation' scheme was proved to exist.

This paper is organized as follows. In section 2, we set our framework and state the main results.
Section 3 is devoted to prove the Theorem 1 which states existence, uniqueness and regularity results of a solution to the discretized scheme.
Finally, in section 4, we prove the Theorem 2, namely the existence and the regularity of the global attractor.

2 The framework and main results

In this section, we review the notations used throughout the article, and we refer to some mathematical tools, which are very useful to the success of our discussion. We set $\Omega = \mathbb{T}^2$, and

let $L^p(\Omega)$ denotes the space of the pth-power integrable functions normed by

$$\| f \|_p = (\int |f(x)|^p dx)^{\frac{1}{p}}$$

for $p \in [1, \infty)$. As usual, $\hat{f}$ is the Fourier transform of f, i.e.

$$\hat{f}(k) = \frac{1}{(2\pi)^2} \int_\Omega f(x) e^{-ik.x} dx.$$

$\Lambda = (-\Delta)^{\frac{1}{2}}$, denotes the pseudo-differential operator given by

$$(\hat{\Lambda f})(k) = |k| \hat{f}(k).$$

More generally, Λf can be identified by means of Fourier series as,

$$\Lambda^\beta f(x) = \sum_{k \in \mathbb{Z}^2} |k|^\beta \hat{f}(k) e^{ik.x}.$$

We define the Sobolev spaces

$$H^{s,p} = H^{s,p}(\Omega) = \{f \in L^p(\Omega), \quad \Lambda^s f \in L^p(\Omega)\}.$$

Since we consider periodic boundary conditions on Ω, obviously, all derivatives of the solution θ are mean zero. Then, $\bar{\theta}$ the mean value of θ satisfies

$$\frac{d}{dt}\bar{\theta} = \frac{1}{|\Omega|}\frac{d}{dt}\int_\Omega \theta dx = \bar{f},$$

where

$$\bar{\theta} = \frac{1}{|\Omega|}\int_\Omega \theta(x) dx, \quad and \quad \bar{f} = \frac{1}{|\Omega|}\int_\Omega f(x) dx.$$

Hence, without loss of generality, we may restrict the discussion to θ that obeys for all time to $\bar{\theta} = 0$. Otherwise, we can replace f with $f - \bar{f}$ and θ with $\theta - \bar{\theta}$ and Eq.(1) will not change essentially. For that purpose we introduce

$$\dot{H}^{s,p} = \dot{H}^{s,p}(\Omega) = \left\{ f \in H^{s,p}, \quad \int_\Omega f(x) dx = 0 \right\}.$$

Accordingly, we introduce the sobolev spaces

$$\dot{H}^s = \dot{H}^{s,2} \qquad \text{and} \quad \dot{L}^p = \dot{H}^{0,p}$$

In order to consider (QG) as a dynamical system, we assume that f is a given time independent scalar function, which belongs at least to $\dot{L}^2$. We assume that the initial data θ_0 belongs to L^2 and satisfies,

$$\frac{1}{|\Omega|}\int_\Omega \theta_0(x) dx = 0, \tag{4}$$

so that the solution θ satisfies also (4).

Now, we are in position to introduce the numerical scheme. Let $\tau > 0$ be a fixed real, and set $t^n = n\tau$ for $n \in \mathbb{N}$. Now we recursively construct elements θ^{n+1} which approches $\theta(t^{n+1})$, by setting:

$$
\frac{\theta^{n+1}-\theta^n}{\tau}+\nu(-\Delta)^\alpha\theta^{n+1}+\nabla.(u^{n+1}\theta^{n+1})=f. \tag{5}
$$

Notice that θ^0 is an approximation of θ_0, and $u^{n+1} = \mathcal{R}^\perp(\theta^{n+1})$.
In our study, we consider the space domain $\Omega=\mathbb{T}^2$ and we follow the guidelines of [12]. Moreover, we take use of the strategy of N. Ju in [11], we quote particularly the improved positivity lemma proved in [11], which is of major utility for our success at this stage, supplemented by the generalized commutator estimate due to Kenig Ponce and Vega [13].
Our main results state as follows :

Theorem 1 *Let $\alpha\in]0,1]$ and $f\in\dot{L}^2$. Then, for all $\theta^n\in\dot{L}^2$ there exists at least one solution θ^{n+1} of (5) which belongs to $\dot{H}^\alpha$. Moreover, if $\frac{2}{3}<\alpha<1$ then $\theta^{n+1}\in\dot{H}^{2\alpha}$ and when $\tau>0$ is small enough, and $\frac{2}{3}\le\alpha<1$ this solution is unique.*

Furthermore, let

$$
H:=\left\{\theta\in\dot{L}^{p_\alpha};\ \|\,\theta\,\|_{p_\alpha}\le M\right\}, \tag{6}
$$

where $M>0$ is conveniently chosen, and consider the map

$$
S:H\to H,\quad\ \theta^n\mapsto\theta^{n+1},
$$

defined by (5). We denote by d the metric distance defined by the $\dot{L}^2$ norm, then we state:

Theorem 2 *Let $\frac{2}{3}<\alpha<1$ and suppose that $f\in\dot{L}^{p_\alpha}$ with $p_\alpha=\frac{2}{1-\alpha}$. Then the map $S:H\to H$ is continuous with respect to the $\dot{L}^2$ topology and defines a discrete dynamical system $(S^n)_n$ on the complete metric space $(H,\,d)$. Besides, $(S^n)_n$ possesses a global attractor $\mathcal{A}$ in H, which is a compact subset in $\dot{H}^\alpha$ and included in $\dot{H}^{2\alpha}$.*

Actually, in order to prove the above results, we enounce different lemmas and inequalities used in the later proofs. Let us start by a technical Lemma, which is a consequence of the Brouwer's lemma [19, p.164]:

Lemma 1 *Let X be a finite dimensional Hilbert space endowed with the inner product $(.\,,\,.)$ and with the corresponding norm $\|\ .\ \|$, and set $F:\ X\to X$ a continuous form that satisfies: $\exists\ R>0$ such that,*

$$
[F(\xi),\xi]\ge 0\quad\textit{for}\ \ |\xi|\le R,
$$

then, there exists $\xi_0\in X$ such that $|\xi_0|\le R$, and $F(\xi_0)=0$.

$\blacksquare$

Next, let us refer to the work of N. Ju, to recall an improved positivity Lemma [11, Lemma 3.3 page 167],

Lemma 2 *Let $p\ge 2,\ s\in[0,2]$, and $\Omega=\mathbb{T}^2$, then suppose that θ the solution of (1) belongs to L^p, and so is $\Lambda^s\theta$. Then we have*

$$
p\int_\Omega|\theta|^{p-2}\theta\Lambda^s\theta\ge 2\int_\Omega(\Lambda^{\frac{s}{2}}|\theta|^{\frac{p}{2}})^2. \tag{7}
$$

$\blacksquare$

The uniform Gronwall lemma presented in Temam [20], is a powerful tool for a priori estimation. We recall a discrete version of the uniform Gronwall lemmas given in Shen [18], which will be useful in our discussion.

Lemma 3 *(***Discrete Uniform Gronwall Lemma.***)*
Let $\Delta t > 0$ and let (f_n),(g_n) and (y_n) be three positive sequences. Suppose that $\exists n_0 \geq 0, \quad r > 0, \quad a_0(r), \quad a_1(r), \quad a_2(r)$ non negative functions such that

$$\frac{y_{n+1} - y_n}{\Delta t} \leq f_n y_n + g_n, \qquad \forall n \geq 0,$$

$$\forall k_0 \geq n_0 \qquad \Delta t \sum_{n=k_0}^{N+k_0} f_n \leq a_0(r); \quad \Delta t \sum_{n=k_0}^{N+k_0} g_n \leq a_1(r); \quad \Delta t \sum_{n=k_0}^{N+k_0} y_n \leq a_2(r),$$

$N = [\frac{r}{\Delta t}]$ an integer, then

$$y_n \leq (a_1 + \frac{a_2}{r}) \exp(a_0) \qquad \forall n \geq n_0 + N.$$

$\blacksquare$

Let us also recall a product estimate in Sobolev spaces, due to Kenig, Ponce and Vega [13],

$$\| \Lambda^s(u\theta) \|_2 \leq c[\| u \|_q \| \Lambda^s\theta \|_p + \| \theta \|_q \| \Lambda^s u \|_p], \tag{8}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, and the Poincaré's inequality,

$$\| \Lambda^\alpha\theta \|_2^2 \quad \geq \quad C_0 \| \theta \|_2^2, \tag{9}$$

where denoting by λ_1 the first nonnegative eigenvalue of the operator $(-\Delta)$, with periodic boundary conditions, then $C_0 = \lambda_1^\alpha$.
Finally, we introduce the Faedo-Galerkin method used for the resolution of nonlinear variational formulations. That is, for any $m \in \mathbb{N}^*$, we consider the finite dimensional subspace of $\dot{H}^1$,

$$V_m = \text{Span}\{e_k = e^{ik.x},\ k = (k_1, k_2); 0 < \max(|k_1|, |k_2|) \leq m\},$$

endowed with the same inner product and the same norm as those of $\dot{H}^1$.
Accordingly, we denote by P_m, the orthogonal projection onto V_m, defined by:

$$\begin{array}{rcl} P_m : \dot{L}^2 & \to & V_m \\ u & \mapsto & P_m(u) = \displaystyle\sum_{\max(|k_1|,|k_2|)\leq m} \hat{u}(k) e_k, \end{array}$$

which commutes with the fractional Laplace operator.

$\blacksquare$

3 Proof of Theorem 1

We shall split the work into three steps : existence, uniqueness and regularity for solutions of (5). To begin with, we prove the first step.

3.1 Existence

We take the inner product of (5) with θ^{n+1} in $\dot{L}^2$. Using periodic boundary conditions together with the fact that u^{n+1} is divergence free, this leads to:

$$\| \theta^{n+1} \|_2^2 \ + \nu\tau \ \| \Lambda^\alpha \theta^{n+1} \|_2^2 = (\tau f + \theta^n, \theta^{n+1}). \tag{10}$$

Thus, by Young's and Cauchy Schwartz's inequalities, we obtain:

$$\| \theta^{n+1} \|_2^2 \ + 2\nu\tau \ \| \Lambda^\alpha \theta^{n+1} \|_2^2 \leq \ 2\tau^2 \ \| f \|_2^2 \ + 2 \| \theta^n \|_2^2 . \tag{11}$$

We infer from the a priori estimate (11), that we shall look for a weak solution θ^{n+1} that belongs to $\dot{H}^\alpha$. On the other hand, by considering a variational formulation of our problem, we remark that nonlinearity of the variational form under consideration prevents us from resolving the equation (5) by Lax-Milgram Lemma in $\dot{H}^\alpha$. Therefore, to contribute to the control of the nonlinearity, we mimic the strategy in [9] for (QG) equation, so we proceed to a variational regularization of (5), which reads:

$$\begin{gathered} for\ \varepsilon > 0, \quad find\ \ \theta_\varepsilon^{n+1} \in \dot{H}^1\ \ such\ \ that, \ \ \forall v \in \dot{H}^1 \\ \tfrac{1}{\tau}(\theta_\varepsilon^{n+1} - \theta^n, v) + \nu(\Lambda^\alpha \theta_\varepsilon^{n+1}, \Lambda^\alpha v) + (\nabla.(u_\varepsilon^{n+1}\theta_\varepsilon^{n+1}), v) \\ +\varepsilon(\nabla\theta_\varepsilon^{n+1}, \nabla v) = (f, v). \end{gathered} \tag{12}$$

with $u_\varepsilon^{n+1} = \ \mathcal{R}^\perp(\theta_\varepsilon^{n+1})$.
The problem (12) is nonlinear and its resolution is based on the Faedo-Galerkin approximation method introduced in Section 2.
To approach θ_ε^{n+1} the solution of (12), we have to solve the following variational problem:

$$\begin{gathered} for\ \varepsilon > 0,\ and\ m \in \mathbb{N}^*, \quad find\ \ \theta_{\varepsilon,m}^{n+1} \in V_m\ \ such\ \ that, \ \ \forall v \in V_m \\ \tfrac{1}{\tau}(\theta_{\varepsilon,m}^{n+1} - P_m\theta^n, v) + \nu(\Lambda^\alpha \theta_{\varepsilon,m}^{n+1}, \Lambda^\alpha v) + (\nabla.(u_{\varepsilon,m}^{n+1}\theta_{\varepsilon,m}^{n+1}), v) \\ +\varepsilon(\nabla\theta_{\varepsilon,m}^{n+1}, \nabla v) = (P_m f, v). \end{gathered} \tag{13}$$

We state and prove the following result:

Proposition 1 *$\forall m \in \mathbb{N}^*$ and $\forall \varepsilon > 0$, there exists $\theta_{\varepsilon,m}^{n+1} \in V_m$ a solution of (13). Moreover, we have the following a priori estimates:*

$$\| \theta_{\varepsilon,m}^{n+1} \|_2 \leq 2K_0, \tag{14}$$

$$\| \Lambda^\alpha \theta_{\varepsilon,m}^{n+1} \|_2 \leq \frac{K_0}{\sqrt{\nu\tau}}, \tag{15}$$

$$\| \nabla \theta_{\varepsilon,m}^{n+1} \|_2 \leq \frac{K_0}{\sqrt{\epsilon\tau}} \tag{16}$$

where $K_0 = \sqrt{\tau^2 \| f \|_2^2 + \| \theta^n \|_2^2 + 1}$.

Proof : in order to prove the existence of such solution, we need the technical Brouwer's Lemma 1. For that purpose, consider here $X = V_m$, and $F : V_m \to V_m$ be defined by

$$F(\theta_m) = \theta_m + \nu\tau\Lambda^{2\alpha}\theta_m + \tau P_m \nabla.(u_m\theta_m) - \tau\varepsilon\Delta\theta_m - \tau P_m f - P_m\theta^n. \tag{17}$$

First of all we shall verify the conditions of Lemma 1, on F defined by (17). Proving the continuity of F is straightforward from the continuity of the operators Λ^α, $\mathcal{R}$, ∇ and P_m on

V_m. Next, taking the $\dot{L}^2$ inner product of $F(\theta_{\varepsilon,m}^{n+1})$ with $\theta_{\varepsilon,m}^{n+1}$, we get by Young inequality:

$$[F(\theta_{\varepsilon,m}^{n+1}),\theta_{\varepsilon,m}^{n+1}] \geq \frac{1}{2} \parallel \theta_{\varepsilon,m}^{n+1} \parallel_2^2 + \nu\tau \parallel \Lambda^\alpha \theta_{\varepsilon,m}^{n+1} \parallel_2^2 + \varepsilon\tau \parallel \nabla\theta_{\varepsilon,m}^{n+1} \parallel_2^2 - K_0^2 \tag{18}$$

$$\geq \frac{1}{2} \parallel \theta_{\varepsilon,m}^{n+1} \parallel_2^2 - K_0^2. \tag{19}$$

Thus, thanks to (19), it becomes clear that for $\theta_{\varepsilon,m}^{n+1} \in V_m$ such that $\parallel \theta_{\varepsilon,m}^{n+1} \parallel_2 = 2K_0$, we have:

$$[F(\theta_{\varepsilon,m}^{n+1}),\theta_{\varepsilon,m}^{n+1}] \geq K_0^2 > 0.$$

Hence by Brouwer's Lemma 1, we obtain the existence of $\theta_{\varepsilon,m}^{n+1} \in V_m$ such that $\parallel \theta_{\varepsilon,m}^{n+1} \parallel_2 \leq 2K_0$, and $F(\theta_{\varepsilon,m}^{n+1}) = 0$. Moreover, (15) and (16) follow immediately from (18). ■

Now we state and prove:

Proposition 2 *For all $\varepsilon > 0$, (12) admits a solution $\theta_\varepsilon^{n+1} \in \dot{H}^1$ that satisfies (14), (15) and (16).*

Proof : obviously, such a result is obtained by getting the limit on m. At first sight, the estimates (14) and (16) infer that $(\theta_{\varepsilon,m}^{n+1})_m$ is bounded in $\dot{H}^1$ then it admits a subsequence still denoted by $(\theta_{\varepsilon,m}^{n+1})_m$ which weakly converges to θ_ε^{n+1} in $\dot{H}^1$, and strongly in $\dot{H}^\alpha$ and in $\dot{L}^4$, owing to the compact Sobolev imbedding

$$\dot{H}^1 \hookrightarrow \dot{H}^\alpha \hookrightarrow \dot{L}^4. \tag{20}$$

Thus, we go back to (13) and we let m goes towards the infinity. Using the continuity of the Riesz operator on $\dot{L}^p$ spaces, we get

$$u_{\varepsilon,m}^{n+1} \rightarrow u_\varepsilon^{n+1} = \mathcal{R}^\perp(\theta_\varepsilon^{n+1}) \quad in\ \dot{L}^4. \tag{21}$$

From (20) we deduce that,

$$\Lambda^\alpha\theta_{\varepsilon,m}^{n+1} \rightarrow \Lambda^\alpha\theta_\varepsilon^{n+1} \ \ in\ \dot{L}^2. \tag{22}$$

The same above arguments yield:

$$< \nabla.(u_{\varepsilon,m}^{n+1}\theta_{\varepsilon,m}^{n+1}), \eta >_{(\dot{H}^{-1},\dot{H}^1)} \rightarrow < \nabla.(u_\varepsilon^{n+1}\theta_\varepsilon^{n+1}), \eta >_{(\dot{H}^{-1},\dot{H}^1)} .$$

Thus, we conclude that θ_ε^{n+1} is a solution of (12) which belongs to $\dot{H}^1$, and the estimates (14), (15) and (16), follow promptly by getting the limits on m.

■

Now we are ready to state the existence of a solution to (5).

Proposition 3 *For all $n \geq 0$, there exists θ^{n+1} solution of (5), which belongs to $\dot{H}^\alpha$.*

Proof: using (14), (15) and (16) on θ_ε^{n+1}, there exists a subsequence still denoted by $(\theta_\varepsilon^{n+1})_{\varepsilon>0}$ such that $\sqrt{\varepsilon}\nabla\theta_\varepsilon^{n+1} \rightharpoonup h$ in $\dot{L}^2$, $\theta_\varepsilon^{n+1} \rightharpoonup \theta^{n+1}$ in $\dot{H}^\alpha$, and $\theta_\varepsilon^{n+1} \rightarrow \theta^{n+1}$ in $\dot{L}^2$ when $\varepsilon \rightarrow 0$. Hence, on the one hand $\varepsilon\Delta\theta_\varepsilon^{n+1} \rightarrow 0$ in $\dot{H}^{-1}$, and on the other hand $u_\varepsilon^{n+1}\theta_\varepsilon^{n+1} \rightarrow u^{n+1}\theta^{n+1}$ in $\dot{L}^2$, so we get $\nabla(u_\varepsilon^{n+1}\theta_\varepsilon^{n+1}) \rightarrow \nabla(u^{n+1}\theta^{n+1})$ in $\dot{H}^{-1}$ when $\varepsilon \rightarrow 0$. Therefore, the proposition is proved.

■

3.2 Regularity

Actually we move to state regularity results. We point out that the regularity of θ_ε^{n+1}, the solution of (12), echoes directly on the regularity of θ^{n+1}. Thus, to begin with, we prove the following proposition, which states some regularity results for θ_ε^{n+1}.

Proposition 4 *$\forall\ \varepsilon\ >0,\ \forall\ n\in\mathbb{N},\ \theta_\varepsilon^{n+1}$ the solution of (12) belongs to $\dot{H}^2$. Furthermore, there exists $C_n>0$ independent of ε such that*

$$\|\theta_\varepsilon^{n+1}\|_{\dot{H}^{2\alpha}}\leq C_n, \tag{23}$$

for all $\alpha>\frac{2}{3}$.

Proof : let B be the operator defined by

$$B:=(I-\varepsilon\tau\Delta)^{-1},$$

which is a regularizing operator of order 2. Then (12) can be rewritten equivalently as

$$\theta_\varepsilon^{n+1}+\tau\nu B(-\Delta)^\alpha\theta_\varepsilon^{n+1}+\tau B\nabla(u_\varepsilon^{n+1}\theta_\varepsilon^{n+1})=B(\tau f+\theta^n). \tag{24}$$

Now, since $f\in\dot{L}^2$, and referring to Eq.(24), then the maximal regularity of θ_ε^{n+1} is $\dot{H}^2$. Therefore, we are going to prove that $\theta_\varepsilon^{n+1}\in\dot{H}^2$ in two steps. Firstly, we recall that the regularity $\dot{H}^1$ for θ_ε^{n+1} is ensured due to results of Proposition 2. Then we remark that thanks to the Sobolev imbeddings, and the continuity of the Riesz operator, we have $u_\varepsilon^{n+1}\theta_\varepsilon^{n+1}\in\dot{L}^p$, for all $p\in[1,\infty)$. Thus,

$$B\nabla(u_\varepsilon^{n+1}\theta_\varepsilon^{n+1})\in\dot{H}^{1,p}. \tag{25}$$

Moreover,

$$B(-\Delta)^\alpha\theta_\varepsilon^{n+1}\ \in\dot{H}^{3-2\alpha}. \tag{26}$$

On the other hand, due to the Sobolev imbeddings, $\dot{H}^{3-2\alpha}\hookrightarrow\dot{H}^{1,\frac{2}{2\alpha-1}}$, when $\alpha<1$, and $\dot{H}^2\hookrightarrow\dot{H}^{1,p}$, $\forall p>2$, we deduce that for $\alpha<1$ $\theta_\varepsilon^{n+1}\in\dot{H}^{1,p_0}$, which is an algebra for some $p_0>2$. Now, by a bootstrap argument and using the Sobolev imbeddings, we deduce that $\theta_\varepsilon^{n+1}\in\dot{H}^2$

Actually, we move on toward the estimate (23). For that purpose, let β be a real that satisfies $0<\beta\leq\alpha$, and that have to be fixed later. We are going to prove that,

$$\|\ \theta_\varepsilon^{n+1}\ \|_{\dot{H}^{\beta+\alpha}}\leq C_n, \tag{27}$$

for all $\alpha>\frac{2}{3}$ and $0<\ \beta\leq 3\alpha-2$.
For that purpose, we take $v=\Lambda^{2\beta}\theta_\varepsilon^{n+1}$ in (12), then we get:

$$\begin{gathered}(\theta_\varepsilon^{n+1},\Lambda^{2\beta}\theta_\varepsilon^{n+1})+\nu\tau(\Lambda^{2\alpha}\theta_\varepsilon^{n+1},\Lambda^{2\beta}\theta_\varepsilon^{n+1})+\tau(\nabla.(u_\varepsilon^{n+1}\theta_\varepsilon^{n+1}),\Lambda^{2\beta}\theta_\varepsilon^{n+1})\\ -\varepsilon\tau(\Lambda^2\theta_\varepsilon^{n+1},\Lambda^{2\beta}\theta_\varepsilon^{n+1})=(\tau f+\theta^n,\Lambda^{2\beta}\theta_\varepsilon^{n+1}),\end{gathered} \tag{28}$$

which leads to:

$$\begin{gathered}\|\ \Lambda^\beta\theta_\varepsilon^{n+1}\ \|_2^2+\nu\tau\ \|\ \Lambda^{\alpha+\beta}\theta_\varepsilon^{n+1}\ \|_2^2+\varepsilon\tau\ \|\ \Lambda^{1+\beta}\theta_\varepsilon^{n+1}\ \|_2^2\\ \leq|(\tau f+\theta^n,\Lambda^{2\beta}\theta_\varepsilon^{n+1})|+\tau|(\nabla.(u_\varepsilon^{n+1}\theta_\varepsilon^{n+1}),\Lambda^{2\beta}\theta_\varepsilon^{n+1})|.\end{gathered} \tag{29}$$

Using Cauchy Schwartz and Young inequalities, together with the embedding

$$\dot{H}^{\alpha+\beta} \hookrightarrow \dot{H}^{2\beta} \quad for \;\; \beta \leq \alpha, \tag{30}$$

we obtain,

$$|(\tau f + \theta^n, \Lambda^{2\beta}\theta_\varepsilon^{n+1})| \leq \frac{C}{\nu\tau} \parallel \tau f + \theta^n \parallel_2^2 + \frac{\nu\tau}{4} \parallel \Lambda^{\beta+\alpha}\theta_\varepsilon^{n+1} \parallel_2^2, \tag{31}$$

and similarly,

$$\tau|(\nabla.(u_\varepsilon^{n+1}\theta_\varepsilon^{n+1}), \Lambda^{2\beta}\theta_\varepsilon^{n+1})| \leq \frac{\tau}{\nu} \parallel \Lambda^{1+\beta-\alpha}(u_\varepsilon^{n+1}\theta_\varepsilon^{n+1}) \parallel_2^2 + \frac{\tau\nu}{4} \parallel \Lambda^{\beta+\alpha}\theta_\varepsilon^{n+1} \parallel_2^2 . \tag{32}$$

In order to estimate the product $u_\varepsilon^{n+1}\theta_\varepsilon^{n+1}$ in $\dot{H}^{1+\beta-\alpha,\,2}$, we shall use the product estimation (8), for $s = 1+\beta-\alpha,\;\; q = \frac{2}{1-\alpha}$, and $p = \frac{2}{\alpha}$. By these data, the Sobolev imbedding

$$\dot{H}^\alpha \hookrightarrow \dot{L}^q \;\; and \;\; \dot{H}^\alpha \hookrightarrow \dot{H}^{s,p}, \tag{33}$$

are satisfied for $\beta \leq 3\alpha - 2$.
Now we make use of the continuity of the Riesz operator on $\dot{L}^p$ to get:

$$\parallel \Lambda^s(u_\varepsilon^{n+1}\theta_\varepsilon^{n+1}) \parallel_2 \leq c \parallel \theta_\varepsilon^{n+1} \parallel_q \parallel \Lambda^s\theta_\varepsilon^{n+1} \parallel_p . \tag{34}$$

Thus, by (33), (15), (31) and (32) we derive the wished bound of θ_ε^{n+1}, on the $\dot{H}^{\beta+\alpha}$ norm. Consequently, we obtain (23) using (27)and a bootstrap argument. ■

Proposition 5 *For* $\alpha \in]0,\;1],\quad \theta^{n+1}$ *the solution defined by Proposition 2 belongs to* $\dot{H}^\alpha$. *Moreover, for* $\alpha \in]\frac{2}{3}, 1[$, *this solution belongs to* $\dot{H}^{2\alpha}$.

Proof : since the estimates (14) and (23) are independent of ε, then by making $\varepsilon \to 0$, we obtain the desired result. ■

3.3 Uniqueness

Proposition 6 *Let* $\theta^n \in \dot{L}^2$ *and* $\alpha \in \;[\frac{2}{3},\,1[$. *Then, for* $\tau > 0$ *small enough,* θ^{n+1} *the solution of (5) is unique.*

Proof : let $\theta_1^n,\; \theta_2^n \in \dot{L}^2$ and consider $\theta_1^{n+1},\; \theta_2^{n+1} \in \dot{H}^\alpha$ the respective solutions according to (5).
Now, set $\theta^n = \theta_2^n - \theta_1^n$, $\theta^{n+1} = \theta_2^{n+1} - \theta_1^{n+1}$, and $u^{n+1} = \mathcal{R}^\perp\theta^{n+1}$. Then, θ^{n+1} satisfies,

$$\theta^{n+1} - \theta^n + \nu\tau(-\Delta)^\alpha\theta^{n+1} + \tau\nabla(u^{n+1}\theta_2^{n+1} + u_1^{n+1}\theta^{n+1}) = 0. \tag{35}$$

Taking the inner product of (35) with θ^{n+1}, we find:

$$\parallel \theta^{n+1} \parallel_2^2 - \parallel \theta^n \parallel_2^2 + 2\nu\tau \parallel \Lambda^\alpha\theta^{n+1} \parallel_2^2 \leq | \underbrace{2\tau \int \nabla(u^{n+1}\theta_2^{n+1})\theta^{n+1}}_{I_{n+1}} | . \tag{36}$$

Now, by Cauchy Schwartz inequality, we find

$$|I_{n+1}| \leq 2\tau \parallel \Lambda^{1-\beta}(u^{n+1}\theta_2^{n+1}) \parallel_2 \parallel \Lambda^\beta\theta^{n+1} \parallel_2, \tag{37}$$

for some $\beta \in [\frac{2}{3},\ \alpha[$. Then, using the product estimation (8), with $s = 1 - \beta$, $\frac{1}{q} = 1 - \beta$ and $\frac{1}{p} = \beta - \frac{1}{2}$, supplemented by the compact Sobolev Imbedding $\dot{H}^{\alpha} \hookrightarrow \dot{H}^{\beta}$, and the continuity of the Riesz operator in $\dot{L}^p$ spaces yield:

$$|I_{n+1}| \leq 2C\tau \parallel \theta_2^{n+1} \parallel_{\dot{H}^{\alpha}} \parallel \Lambda^{\beta}\theta^{n+1} \parallel_2^2 . \tag{38}$$

Moreover, by interpolation we get:

$$\parallel \Lambda^{\beta}\theta^{n+1} \parallel_2^2 \leq C \parallel \theta^{n+1} \parallel_2^{2(1-\frac{\beta}{\alpha})} \parallel \Lambda^{\alpha}\theta^{n+1} \parallel_2^{2\frac{\beta}{\alpha}} . \tag{39}$$

Now, using the Young Inequality and inserting (39) in (38), we deduce that,

$$|I_{n+1}| \leq \frac{1}{2} \parallel \theta^{n+1} \parallel_2^2 + C\tau^2 \parallel \theta_2^{n+1} \parallel_{\dot{H}^{\alpha}}^2 \parallel \Lambda^{\alpha}\theta^{n+1} \parallel_2^2 . \tag{40}$$

Replacing (40) in (36), we deduce that there exists $\tilde{C} > 0$ independent of n such that

$$\parallel \theta^{n+1} \parallel_2^2 + \tau[2\nu - \tilde{C}\tau \parallel \theta_2^{n+1} \parallel_{\dot{H}^{\alpha}}^2] \parallel \Lambda^{\alpha}\theta^{n+1} \parallel_2^2 \leq 2 \parallel \theta^n \parallel_2^2 . \tag{41}$$

Therefore, for $\tau > 0$ small enough such that

$$\tau \parallel \theta_2^{n+1} \parallel_{\dot{H}^{\alpha}}^2 \leq \frac{2\nu}{\tilde{C}}, \tag{42}$$

we infer from (41) that,

$$\parallel \theta^{n+1} \parallel_2^2 \leq 2 \parallel \theta^n \parallel_2^2 \tag{43}$$

which yields the uniqueness of θ^{n+1} the solution of (5).

■

4 Proof of Theorem 2

We move to prove the existence and the regularity of the global attractor. We recall that Theorem 1 provides us a semi-discrete dynamical system, $(\dot{L}^2, (S^n)_{n\in\mathbb{N}})$ for $\alpha \geq \frac{2}{3}$, for τ small enough, that is given by mean of the following map

$$\begin{aligned} S:\ \ \dot{L}^2 &\to \dot{L}^2 \\ \theta^n &\mapsto S\theta^n = \theta^{n+1}, \end{aligned}$$

where θ^{n+1} is the unique solution of (5), when $\alpha \geq \frac{2}{3}$. Notice that following recursively Eq.(5), and starting from θ^0, we define the operator $S^n :\ \dot{L}^2 \to\ \dot{L}^2$ such that $S^n\theta^0 = \theta^n$.
To go ahead, it is well known that to describe the long time behavior of solutions to the so defined dynamical system, we shall concentrate on the dynamics of some absorbing sets for the semi-group introduced above. We recall that general results concerning the existence of global attractors are given in the book of R. Temam [19, Chapter 1] for both continuous and discrete dynamical systems. To get the existence of the global attractor, we have to fulfill the conditions of the following proposition:

Proposition 7 *Let H be a Hilbert or complete metric space and let $S\ :\ H \to\ H$ be a continuous map, that satisfies the following properties :*

1. *there exists a bounded absorbing set* $\mathcal{B} \subset H$, *such that*

$$\forall \theta^0 \in H,\ \exists n_0(\theta_0), \quad \forall n \geq n_0(\theta_0), \quad S^n\theta_0 \ \in\ \mathcal{B} \tag{44}$$

2. S^n *is uniformly compact for* n *large enough. It means that for every bounded set* $B \subset H$, *the set* S^nB *is relatively compact in* H.

Then, there exists an invariant compact set $\mathcal{A} \subset H$, *that attracts all trajectories* $S^n\theta^0$, *for all* $\theta^0 \in H$. *More precisely,*

$$S^n(\mathcal{A}) = \mathcal{A}, \qquad \text{and} \quad dist(S^n\theta_0, \mathcal{A}) \to 0, \quad \text{when} \quad n \to \infty.$$

Hence, $\mathcal{A} = \omega(\mathcal{B})$, *the* $\omega-$*limit set of* $\mathcal{B}$, *is the global attractor for the semi-group* $(S^n)_n$. ■

Remark 1 *We notice that to verify the second condition proposed in Proposition 7, we can show that the set* $S^n\mathcal{B}$ *is bounded in a space compactly imbedded in* H. ■

To begin with, we prove the existence of some absorbing sets.

4.1 The $\dot{L}^2$ absorbing set

Proposition 8 *Let* $\tau \in]0,1[$, *and let* $f \in \dot{L}^2$. *Consider* M *a real that satisfies* $M > M_0$, *where* M_0 *is defined by (54). Then the set*

$$E = \{\theta \in \dot{L}^2;\ \parallel \theta \parallel_2 \leq M\}, \tag{45}$$

is an absorbing set positively invariant for S, *that is, for all* $\theta^0 \in \dot{L}^2$ *there exists* $n_0 > 0$ *such that*

$$\forall n \geq n_0, \quad S^n\theta^0 \in E, \tag{46}$$

and

$$S(E) \subset E. \tag{47}$$

Proof: taking the $\dot{L}^2$ inner product of (5) with θ^{n+1} leads to:

$$\parallel \theta^{n+1} \parallel_2^2 - \parallel \theta^n \parallel_2^2 + \parallel \theta^{n+1} - \theta^n \parallel_2^2 + 2\nu\tau \parallel \Lambda^\alpha \theta^{n+1} \parallel_2^2 = 2\tau(f, \theta^{n+1}). \tag{48}$$

Now, thanks to Cauchy-Schwartz and Young inequalities, we find,

$$|\ (\ f, \theta^{n+1}\ )\ | \leq \frac{\parallel f \parallel_2^2}{2\nu C_0} + \frac{C_0\nu}{2} \parallel \theta^{n+1} \parallel_2^2. \tag{49}$$

Hence, inserting (49) in the right hand side of (48), and using the fact that $\parallel \theta^{n+1} - \theta^n \parallel_2^2$ is a positif term, we get:

$$(1 + \nu\tau C_0) \parallel \theta^{n+1} \parallel_2^2 + 2\nu\tau \parallel \Lambda^\alpha \theta^{n+1} \parallel_2^2 \leq \parallel \theta^n \parallel_2^2 + \tau \frac{\parallel f \parallel_2^2}{\nu C_0}. \tag{50}$$

So using the Poincaré inequality (9), we obtain from (50),

$$(1 + \nu\tau C_0) \parallel \theta^{n+1} \parallel_2^2 \leq \parallel \theta^n \parallel_2^2 + \frac{\tau \parallel f \parallel_2^2}{\nu C_0}. \tag{51}$$

Now, we set $r = \frac{1}{1+\nu\tau C_0}$, and we rewrite (51) as follows:

$$\| \theta^{n+1} \|_2^2 \quad \leq \quad r \| \theta^n \|_2^2 + r\frac{\tau}{\nu C_0} \| f \|_2^2 . \tag{52}$$

Then, by a simple induction, we get recursively from (52):

$$\| \theta^n \|_2^2 \quad \leq \quad r^n \| \theta^0 \|_2^2 + (1-r^n)\frac{(1+\nu\tau C_0) \| f \|_2^2}{\nu^2 C_0^2}. \tag{53}$$

We point out that $r < 1$, thus, setting

$$M_0^2 = \frac{(1+\nu C_0) \| f \|_2^2}{\nu^2 C_0^2}, \tag{54}$$

we conclude that for $M > M_0$, the set E defined by (45) satisfies (46).
Moreover E satisfies (47). Indeed, let θ^n belongs to E, then

$$\| \theta^n \|_2^2 \leq M^2.$$

On the other hand, since $M_0 < M$, and

$$\frac{\tau}{\nu C_0} \| f \|_2^2 \leq \nu\tau C_0 M_0^2,$$

(52) leads to:

$$\| \theta^{n+1} \|_2^2 \quad \leq \quad r(1+\nu\tau C_0)M^2 \tag{55}$$
$$= \quad M^2. \tag{56}$$

Thus we obtain (47). ■

4.2 The $\dot{L}^p$ bounded absorbing set

Now, let us prove that for $p > 2$, we get a uniform boundedness on $\| \theta^{n+1} \|_p$. Moreover, we show that there exists an absorbing ball for θ^{n+1} in the $\dot{L}^p$ space.

Proposition 9 *Let $2 < p \leq \frac{2}{1-\alpha}$, and $f \in \dot{L}^p$. We have for all $n \geq 1$,*

$$\| \theta^{n+1} \|_p \leq \frac{1}{1+\frac{2}{p}\nu\tau C_0}(\| \theta^n \|_p + \tau \| f \|_p). \tag{57}$$

Moreover, the set

$$\mathcal{G} = \{\theta \in \dot{L}^{p_\alpha} \ ; \ \| \theta \|_{p_\alpha} \leq M\}, \tag{58}$$

is a bounded absorbing set for S, where $p_\alpha = \frac{2}{1-\alpha}$ and $M > M_1$ where

$$M_1 \ = \ \frac{1}{(1-\alpha)\nu C_0} \| f \|_{p_\alpha} . \tag{59}$$

That is, for all $\theta^0,\ f \ \in \dot{L}^{p_\alpha}$, there exists $n_1 > 0$ such that, $\forall n \geq n_1$

$$S^n\theta^0 \in \mathcal{G}. \tag{60}$$

Proof: suppose that $p > 2$, then we proceed as in [11, page 172]. Our aim is to show for a given (fixed) θ^0, that $\| \theta^{n+1} \|_p$ is also uniformly bounded for $t > 0$.
By taking the inner product of (5) with $p|\theta^{n+1}|^{p-2}\theta^{n+1}$ in $\dot{L}^2$, we get:

$$p \| \theta^{n+1} \|_p^p - p \int_\Omega \theta^n |\theta^{n+1}|^{p-2}\theta^{n+1} + \nu\tau p \int_\Omega \Lambda^{2\alpha}\theta^{n+1}|\theta^{n+1}|^{p-2}\theta^{n+1} + \\ p\tau \int_\Omega u^{n+1}.\nabla\theta^{n+1}|\theta^{n+1}|^{p-2}\theta^{n+1} = p\tau \int_\Omega f|\theta^{n+1}|^{p-2}\theta^{n+1}. \tag{61}$$

By an integration by parts and using the fact that $\nabla.u^{n+1} = 0$, we get

$$p \int_\Omega u^n.\nabla\theta^{n+1}|\theta^{n+1}|^{p-2}\theta^{n+1} = -p \int_\Omega \nabla.u^{n+1}|\theta^{n+1}|^p = 0. \tag{62}$$

Remark 2 *In order to apply the improved positivity Lemma, we have to consider $\theta^{n+1}_{\varepsilon,m}$ defined by (13), instead of θ^{n+1}, since it satisfies the required assumptions of Lemma 2. Hence, all the computations are made formally on θ^{n+1}, however they are valid using the formulation (13), as it is observed earlier, and we conclude by taking the limit on ε.*

Now, we shall use the improved positivity Lemma 2, and we focus particularly on Eq.(7), to get:

$$p \int_\Omega |\theta^{n+1}|^{p-2}\theta^{n+1}\Lambda^{2\alpha}\theta^{n+1} \geq 2 \int_\Omega (\Lambda^\alpha |\theta^{n+1}|^{\frac{p}{2}})^2. \tag{63}$$

On the other hand, thanks to the spectral properties of the operator Λ, we have

$$\int_\Omega (\Lambda^\alpha |\theta^{n+1}|^{\frac{p}{2}})^2 \geq C_0 \| \theta^{n+1} \|_p^p . \tag{64}$$

Now gathering (62), (63) and (64) in (61) we obtain after using the Hölder inequality and simplifying by $\| \theta^{n+1} \|_p^{p-1}$ that,

$$(p + \frac{2}{p}\nu\tau C_0) \| \theta^{n+1} \|_p \leq \| \theta^n \|_p + \tau \| f \|_p . \tag{65}$$

Thus,

$$(1 + \frac{2}{p}\nu\tau C_0) \| \theta^{n+1} \|_p \leq \| \theta^n \|_p + \tau \| f \|_p . \tag{66}$$

We set,

$$K = \frac{1}{1 + \frac{2}{p}\nu\tau C_0}. \tag{67}$$

By a simple induction on (66) we infer that,

$$\begin{aligned} \| \theta^n \|_p &\leq K^n \| \theta^0 \|_p + \tau \sum_{k=1}^n K^k \| f \|_p, \\ &\leq K^n \| \theta^0 \|_p + (1 - K^n)\frac{p}{2\nu C_0} \| f \|_p . \end{aligned} \tag{68}$$

We set,

$$\tilde{M} = \frac{p}{2\nu C_0} \parallel f \parallel_p . \tag{69}$$

Then, for $n \geq n_1(\parallel \theta^0 \parallel_p)$, we'll get the uniform boundedness of $\parallel \theta^n \parallel_p$ independently of θ^0. Thus, since $M > M_1 \geq \tilde{M}$, then the set

$$F = \Big\{ \theta \in \dot{L}^p, \ \parallel \theta \parallel_p \leq M \Big\} \tag{70}$$

is absorbing and positively invariant by S. ■

4.3 The $\dot{H}^\alpha$ absorbing set

Now, to fulfill the second condition of Proposition 7, we have to prove the following result:

Proposition 10 *Let $f \in \dot{L}^{p_\alpha}$, and $N > 0$ an integer, then we set $r = N\tau$. Consider $M > M_2$ where*

$$M_2 = (\frac{r}{\nu} \parallel f \parallel_2^2 + \frac{a_2}{r}) \exp(\frac{rC}{1 - \tau r C}), \tag{71}$$

and a_2 is given by (84), then the set

$$\mathcal{B} \ = \ \{\theta \in E; \ \parallel \Lambda^\alpha \theta \parallel_2 \leq M\}, \tag{72}$$

is a bounded absorbing set for S.

Proof : taking the $\dot{L}^2$ inner product of (5) with $\Lambda^{2\alpha}\theta^{n+1}$ leads to:

$$\begin{aligned} &\tfrac{1}{2\tau}[\parallel \Lambda^\alpha \theta^{n+1} \parallel_2^2 - \parallel \Lambda^\alpha \theta^n \parallel_2^2 + \parallel \Lambda^\alpha(\theta^{n+1} - \theta^n) \parallel_2^2] + \nu \parallel \Lambda^{2\alpha}\theta^{n+1} \parallel_2^2 \\ &\qquad = \textstyle\int_\Omega f \Lambda^{2\alpha}\theta^{n+1} + \int_\Omega \nabla(u^{n+1}\theta^{n+1})\Lambda^{2\alpha}\theta^{n+1}. \end{aligned} \tag{73}$$

At first, We estimate the first term in the right hand side of (73), using Cauchy Schwartz and Young Inequalities. Thus we obtain:

$$| \int_\Omega f \Lambda^{2\alpha}\theta^{n+1} | \leq \frac{\parallel f \parallel_2^2}{\nu} + \frac{\nu}{4} \parallel \Lambda^{2\alpha}\theta^{n+1} \parallel_2^2 . \tag{74}$$

Secondly, we shall estimate the nonlinear part of (73). Namely, For some $0 < \beta \leq \alpha$ we have,

$$| \int_\Omega \nabla(u^{n+1}\theta^{n+1})\Lambda^{2\alpha}\theta^{n+1} | \leq C \parallel \Lambda^{1+\alpha-\beta}(u^{n+1}\theta^{n+1}) \parallel_2 \parallel \Lambda^{\alpha+\beta}\theta^{n+1} \parallel_2 . \tag{75}$$

Actually, we take use again of the product estimate (8), so we get for $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$

$$\parallel \Lambda^{1+\alpha-\beta}(u^{n+1}\theta^{n+1}) \parallel_2 \leq C[\parallel \Lambda^{1+\alpha-\beta}u^{n+1} \parallel_p \parallel \theta^{n+1} \parallel_q + \parallel \Lambda^{1+\alpha-\beta}\theta^{n+1} \parallel_p \parallel u^{n+1} \parallel_q]. \tag{76}$$

Using the continuity of the Riesz operator on L^p spaces for $1 < p < \infty$, and the sobolev imbedding

$$\dot{H}^{\alpha+\beta} \subset \dot{H}^{1+\alpha-\beta,p}, \tag{77}$$

for $\beta = 1 - \frac{1}{p}$, and for $\frac{1}{q} = \beta - \frac{1}{2}$, we deduce that:

$$\| \Lambda^{1+\alpha-\beta}(u^{n+1}\theta^{n+1}) \|_2 \leq C \| \theta^{n+1} \|_q \| \Lambda^{\alpha+\beta}\theta^{n+1} \|_2 . \tag{78}$$

Now, using the $\dot{L}^q$ uniform boundedness in Proposition 9, having $2 < q \leq \frac{2}{1-\alpha}$, we deduce that

$$| \int_\Omega \nabla(u^{n+1}\theta^{n+1})\Lambda^{2\alpha}\theta^{n+1} | \leq C \| \Lambda^{\alpha+\beta}\theta^{n+1} \|_2^2 . \tag{79}$$

Having $0 < \beta \leq \alpha$, an easy interpolation yields:

$$\| \Lambda^{\alpha+\beta}\theta^{n+1} \|_2 \leq \| \Lambda^{\alpha}\theta^{n+1} \|_2^{1-\frac{\beta}{\alpha}} \| \Lambda^{2\alpha}\theta^{n+1} \|_2^{\frac{\beta}{\alpha}}, \tag{80}$$

such that, by Young inequality we obtain

$$| \int_\Omega \nabla(u^{n+1}\theta^{n+1})\Lambda^{2\alpha}\theta^{n+1} | \leq C \| \Lambda^{\alpha}\theta^{n+1} \|_2^2 + \frac{\nu}{4} \| \Lambda^{2\alpha}\theta^{n+1} \|_2^2 . \tag{81}$$

Then, we get by inserting (81) and (74) in (73):

$$\begin{aligned} \frac{1}{\tau}[\| \Lambda^{\alpha}\theta^{n+1} \|_2^2 &- \| \Lambda^{\alpha}\theta^{n} \|_2^2] \\ &+ \nu \| \Lambda^{2\alpha}\theta^{n+1} \|_2^2 \leq C \| \Lambda^{\alpha}\theta^{n+1} \|_2^2 + C_1. \end{aligned} \tag{82}$$

By Proposition 8, we get

$$\tau \sum_{n=n_0}^{N+n_0} \| \Lambda^{\alpha}\theta^{n+1} \|_2^2 \leq a_2, \tag{83}$$

where

$$a_2 = \frac{r}{\nu C_0} \| f \|_2^2 + r\nu C_0 M^2. \tag{84}$$

On the other hand, a little care to Eq. (82) gives:

$$\frac{1}{\tau}[\| \Lambda^{\alpha}\theta^{n+1} \|_2^2 - \| \Lambda^{\alpha}\theta^{n} \|_2^2] \leq \frac{C}{1-\tau C} \| \Lambda^{\alpha}\theta^{n} \|_2^2 + \frac{C_1}{1-\tau C}. \tag{85}$$

We emphasize that since $0 < \tau = O(1)$, we can ensure that $1 - \tau C > 0$, and that owing to the inequalities (81), (78), (58) and the definition of M_1, then $C = C(C_0, \nu).M_1$.
Now thanks to the uniform Gronwall Lemma 3, we get the uniform boundedness of $\| \Lambda^{\alpha}\theta^{n} \|_2^2$,

$$\| \Lambda^{\alpha}\theta^{n+1} \|_2^2 \leq (\frac{r}{\nu} \| f \|_2^2 + \frac{a_2}{r}) \exp(\frac{rC}{1-\tau rC}), \qquad \forall n \geq n_0 + N, \tag{86}$$

and hence we get the existence of $\mathcal{B}$.

∎

4.4 Existence and regularity of the global attractor

Before proceeding further to apply Proposition 7, let us reorder the previous results to depict the convenient phase space H, that will allows us to fulfill the Proposition's conditions.
We fix $M > \max(M_0, M_1, M_2)$ where M_0, M_1 and M_2 are defined by (54), (59) and (71) and consider the set

$$H = \left\{ \theta \in \dot{L}^{p_\alpha};\ \| \theta \|_{p_\alpha} \leq M \right\}. \tag{87}$$

Then, by the definition of M, and owing to Proposition 8, Proposition 9, and Proposition 10, there exists $n \geq max(n_0 + N,\ n_1)$ such that if $\theta^n \in H$ then $\theta^{n+1} = S\theta^n \ \in H$. Hence, we have

$$S : H \to H$$

is well defined. We define $(H,\ d)$ as a complete metric space endowed with the metric d defined by the $\dot{L}^2$ norm. It remains to prove the continuity of S on $(H,\ d)$. Therefore, we state the following Lemma:

Proposition 11 *S is a continuous map from H to H for $\alpha > \frac{2}{3}$.*

Proof: let $\theta_1^n,\ \theta_2^n \in H$ such that $\theta_1^{n+1} = S\theta_1^n$ and $\theta_2^{n+1} = S\theta_2^n$.
We set $\theta^{n+1} = \theta_2^{n+1} - \theta_1^{n+1}$ and $u^{n+1} = \mathcal{R}^\perp \theta^{n+1}$. Then θ^{n+1} satisfies:

$$\theta^{n+1} + \tau B \nabla (u^{n+1}\theta_2^{n+1} + u_1^{n+1}\theta^{n+1}) = B\theta^n, \tag{88}$$

where B is the linear operator defined by $B := (I + \nu\tau(-\Delta)^\alpha)^{-1}$, moreover, we recall that for τ small enough, we have

$$\forall s_1 < s_2,\ \|(1 + \nu\tau(-\Delta)^\alpha)^{-1}\|_{\mathcal{L}(H^{s_1}, H^{s_2})} \leq \frac{c}{\tau^{\frac{s_2 - s_1}{2\alpha}}}, \tag{89}$$

particularly, we have:

$$\begin{aligned} \| B \|_{\mathcal{L}(\dot{L}^2, \dot{L}^2)} &\leq 1, & (90)\\ \| B \|_{\mathcal{L}(\dot{H}^t, \dot{H}^1)} &\leq \frac{C}{\tau^{\frac{1-t}{2\alpha}}}, & (91) \end{aligned}$$

for all $t < 0$ to be fixed later.
Thus we check the following estimations:

$$\begin{aligned} \| \theta^{n+1} \|_2 &\leq \| B \|_{\mathcal{L}(\dot{L}^2, \dot{L}^2)} \| \theta^n \|_2 + \tau \| B(u^{n+1}\theta_2^{n+1}) \|_{\dot{H}^1} \\ &+ \tau \| B(u_1^{n+1}\theta^{n+1}) \|_{\dot{H}^1} \\ &\leq \| \theta^n \|_2 + C\tau^{1 - \frac{1-t}{2\alpha}} [\| u^{n+1}\theta_2^{n+1} \|_{\dot{H}^t} + \| u_1^{n+1}\theta^{n+1} \|_{\dot{H}^t}]. \end{aligned}$$

Now, choosing $q = \frac{3}{2} < 2$ and $t = \frac{q-2}{q} = -\frac{1}{3}$ such that $\dot{L}^q \hookrightarrow \dot{H}^t$, and using the continuity of the Riesz operator and Hölder inequality we obtain:

$$\| \theta^{n+1} \|_2 \leq \| \theta^n \|_2 + C\tau^{1 - \frac{1-t}{2\alpha}} [\| \theta_2^{n+1} \|_{\dot{L}^6} + \| \theta_1^{n+1} \|_{\dot{L}^6}] \| \theta^{n+1} \|_{\dot{L}^2},$$

since $\frac{1}{q} = \frac{1}{2} + \frac{1}{6}$.
Then, using the uniform boundedness of $\| \theta_2^{n+1} \|_{\dot{L}^6}$ and $\| \theta_1^{n+1} \|_{\dot{L}^6}$ owing to Proposition 9, the defined set F given by (70), and the fact that $6 \in]2,\ p_\alpha]$ for $\alpha \geq \frac{2}{3}$, we get:

$$(1 - CM\tau^{1 - \frac{1-t}{2\alpha}}) \| \theta^{n+1} \|_2 \leq \| \theta^n \|_2 .$$

Remark 3 *At this stage, we must emphasize that* $1-\frac{1-t}{2\alpha}>0$ *only for* $\alpha>\frac{2}{3}$*, and hence for* $\tau>0$ *small enough, we get*

$$CM\tau^{1-\frac{1-t}{2\alpha}}<1.$$

This makes end to this proof.

■

Remark 4 *Notice that the result of Proposition 11, yields the uniqueness of* θ^{n+1} *solution of (5), on* H*. Hence, this gives rise to the dynamical system* $(H,\ (S^n)_{n\geq 0})$*.*

On the other hand, S^n satisfies:

Proposition 12 $(S^n)_{n\in\mathbb{N}}$ *is uniformly compact in* H*.*

Proof: since $\mathcal{B}$ is a compact subset in H then Proposition 10 achieves the proof.

■

Therefore, the following main result is proved.

Proposition 13 *For* $\alpha>\frac{2}{3}$*, the dynamical system* $(H,(S^n)_{n\in\mathbb{N}})$*, admits a global attractor* $\mathcal{A}$*, which is included in* $\dot{H}^{2\alpha}$*.*

Proof: the assumptions of Proposition 7 are satisfied thanks to the Propositions 10, 11, and 12. Thus there exists an invariant compact set $\mathcal{A}$ included in H, which is the global attractor

$$\mathcal{A}=\omega(\mathcal{B})=\bigcap_n\overline{\bigcup_{k\geq n}S^k\mathcal{B}}^H, \tag{92}$$

the ω-limit set of $\mathcal{B}$, and where the closure in (92) is taken with respect to the $\dot{L}^2$ metric

Regularity of this attractor have to be ensured from the invariance property $S\mathcal{A}=\mathcal{A}$, and regularity results of subsection 3.2.

■

Furthermore, we state and prove the following result:

Proposition 14 $\mathcal{A}$ *is a compact set in* $\dot{H}^\alpha$*.*

Proof: to prove the compactness of the attractor, we rely on the J. Ball argument [1]. We proceed as follows: let $(\theta_j^{n+1})_j$ a sequence of points of $\mathcal{A}\subset\dot{H}^\alpha$ and $u_j^{n+1}=\mathcal{R}^\perp(\theta_j^{n+1})$. Now, consider the sequence $(\theta_j^n)_j$ such that

$$\theta_j^n=S^n\theta_j,$$

or equivalently,

$$\frac{\theta_j^{n+1}-\theta_j^n}{\tau}+\nu(-\Delta)^\alpha\theta_j^{n+1}+\nabla.(u_j^{n+1}\theta_j^{n+1})=f. \tag{93}$$

Where $\theta_j^0=\theta_j$.

At this stage we consider n_0 such that $\forall n\geq n_0,\ \theta_j^n\in\mathcal{B}_\alpha$.

We emphasize that referring to the previous results and subsections, there exist subsequences still denoted by $(\theta_j^n)_j$ and $(\theta_j^{n+1})_j$, such that

$$\theta_j^n \to \theta^n \quad \text{and} \quad \theta_j^{n+1} \to \theta^{n+1} \quad \text{weakly in} \quad \dot{H}^\alpha, \tag{94}$$

$$\theta_j^n \to \theta^n \quad \text{and} \quad \theta_j^{n+1} \to \theta^{n+1} \quad \text{strongly in} \quad \dot{L}^p, \tag{95}$$

for any $p \in [2, \frac{2}{1-\alpha}]$. Hence, the limits θ^n and θ^{n+1} satisfy:

$$\frac{\theta^{n+1} - \theta^n}{\tau} + \nu(-\Delta)^\alpha \theta^{n+1} + \nabla.(u^{n+1}\theta^{n+1}) = f. \tag{96}$$

We aim to prove that the convergence holds strongly in $\dot{H}^\alpha$.

Let $w_j^n = \theta_j^n - \theta^n$ and $r_j^n = \mathcal{R}^\perp w_j^n$. By subtracting (96) from (93) we find

$$\frac{w_j^{n+1} - w_j^n}{\tau} + \nu(-\Delta)^\alpha w_j^{n+1} + \nabla.(r_j^{n+1}\theta_j^{n+1} + u^{n+1}w_j^{n+1}) = 0. \tag{97}$$

Taking the $\dot{L}^2$ inner product of (97) with w_j^{n+1} we obtain,

$$\begin{aligned} \frac{1}{\tau}(\| w_j^{n+1} \|_2^2 - \| w_j^n \|_2^2) \quad &+ \quad 2\nu \| \Lambda^\alpha w_j^{n+1} \|_2^2 \\ &\leq \quad 2\underbrace{\left| \int \nabla.(r_j^{n+1}\theta_j^{n+1}) w_j^{n+1} \right|}_{I_j^{n+1}}. \end{aligned} \tag{98}$$

Let $\frac{2}{3} < \beta < \alpha$, such that we get

$$\dot{H}^{2\beta-1} \subset \dot{H}^{1-\beta}. \tag{99}$$

By the Cauchy-Schwartz inequality, we have

$$I_j^{n+1} \leq \| \Lambda^{1-\beta}(r_j^{n+1}\theta_j^{n+1}) \|_2 \| \Lambda^\beta w_j^{n+1} \|_2 . \tag{100}$$

Then, thanks to the Sobolev imbedding (99), we get:

$$I_j^{n+1} \leq \| \Lambda^{2\beta-1}(r_j^{n+1}\theta_j^{n+1}) \|_2 \| \Lambda^\beta w_j^{n+1} \|_2 . \tag{101}$$

Now using the pointwise product estimate in Sobolev spaces $\dot{H}^s(\mathbb{T}^2)$, for
$0 < s < 1$, we get

$$I_j^{n+1} \leq [\| \Lambda^\beta r_j^{n+1} \|_2 \| \Lambda^\beta \theta_j^{n+1} \|_2] \| \Lambda^\beta w_j^{n+1} \|_2, \tag{102}$$

which using the continuity of the Riesz operator, and the uniform boundedness results of Proposition 10, yields:

$$I_j^{n+1} \leq C \| \Lambda^\beta w_j^{n+1} \|_2^2 . \tag{103}$$

A simple interpolation of $\dot{H}^\beta$ in $\dot{L}^2$ and $\dot{H}^\alpha$, together with the Young inequality lead to:

$$I_j^{n+1} \leq C \| w_j^{n+1} \|_2^2 + \nu \| \Lambda^\alpha w_j^{n+1} \|_2^2 . \tag{104}$$

Now using the fact that the sequences $(w_j^n)_j$ and $(w_j^{n+1})_j$ converge to 0 in $\dot{L}^2$ we obtain that $(w_j^{n+1})_j$ converges to 0 in $\dot{H}^\alpha$. Hence we get the compactness of $\mathcal{A}$ in $\dot{H}^\alpha$. ■

References

[1] J. Ball. *G*lobal attractors for damped semilinear wave equations. *Partial differential equations and applications, Discrete Contin. Dyn. Syst.*, no. 1-2, 31-52, 10 (2004).

[2] L. Berselli. *V*anishing viscosity limit and long-time behavior for 2D quasi-geostrophic equations. *Indiana Univ. Math. J. 51, No. 4*, 905-930 (2002).

[3] L. Caffarelli and A. Vasseur. *D*rift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Available at http://www.arxiv.org/abs/math/0608447.* (2006).

[4] D. Chae and J. Lee. *G*lobal well-posedness in the super-critical dissipative quasi-geostrophic equations.. *Commun. Math. Phys.*, 233:297-311 (2003).

[5] P. Constantin, D. Cordoba and J. Wu. *O*n the critical dissipative quasi-geostrophic equation. *Indiana Univ. Math. J.*, 50 Spec. Iss.: 97-107 (2001).

[6] P. Constantin, M.C. Lai, R. Sharma, Y.H. Tseng and J. Wu. *N*ew Numerical Results for the Surface Quasi-Geostrophic Equation. *AMS J. (MOS)* No. 35Q35, 35B65, 65T50, 76M22, 86A10.

[7] P. Constantin, A. Majda and E. Tabak. *F*ormation of strong fronts in the 2-D quasi-geostrophic thermal active scalar. *Nonlinearity 7*, 1495-1533 (1994).

[8] P. Constantin and J. Wu. *B*ehavior of solutions of 2D quasi-geostrophic equations. *Siam J.Math. Anal.* , 30:937-948, (1999).

[9] A. Cordoba and D. Cordoba. *A* maximum Principle applied to quasi-geostrophic equations. *Commun. Math. Phys.*, 249 511-528 (2004).

[10] I. Held, R. Pierrehumbert, S. Garner and K. Swanson. *S*urface quasi-geostrophic dynamics. *J. Fluid Mech.* 282, 1-20 (1995).

[11] N. Ju. *T*he maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations. *Comm. Math. Phys.*, 255, 161-181, (2005).

[12] N. Ju. *O*n the global stability of a temporal discretization scheme for the Navier-Stokes equations. *IMA journal of num. anal* ., 22, pp. 577-597, (2002).

[13] C. Kenig, G. Ponce and L. Vega. *W*ell-posedness of the initial value problem for the Korteweg-De Vries equation. *J. Amer. Math. Soc*, 4, 323-347 (1991).

[14] A. Kiselev, F. Nazarov and A. Volberg. *G*lobal well-posedness for the critical $2D$ dissipative quasi-geostrophic equation. *Invent. Math.,* 167: 445-453, (2007).

[15] J. Pedlosky. *G*eophysical Fluid Dynamics. *New York: Springer-Verlag*, (1987).

[16] S. Resnick. *D*ynamical problems in Non-linear Advances Partial Differential Equations. *Ph.D. thesis, University of Chicago,* II, (1995).

[17] T. Runst and W. Sickel. *S*obolev spaces of Fractional order, Nemytskij Operators, and Nonlinear Partial Differantial Equations. *De Gruyter Series in Nonlinear Analysis And Applications 3.* Walter de Gruyter Berlin-New York, (1996).

[18] J. Shen. *L*ong Time Stabilities and Convergence for the Fully discrete Nonlinear Galerkin Methods. *Appl. Anal., 38, 201-229,* (1990).

[19] R. Temam *N*avier-Stokes Equations. *Studies in Mathematics and its applications, North-Holland*.

[20] R. Temam *I*nfinite Dimensional Dynamical Systems in Mechanics and Physics. *2nd Edition, Berlin: Springer,* (1997).